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FURSTENBERG'S "TIMES 2, TIMES 3" CONJECTURE

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Abstract

Furstenberg conjectured that the Lebesgue measure is the unique non-atomic ergodic probability measure on the circle that is invariant for both the doubling and tripling maps. Furstenberg proved in 1967 that every minimal set is either finite or the whole space, as a special case of his results, which is a topological equivalent of the conjectured result. In 1990, Rudolph proved a partial result, namely, that the Lebesgue measure is the only such measure if we add a condition that the entropy of the measure is positive.

This expository paper illustrates the method described by Kalinin and Katok in 2001, which is used to prove the result of Rudolph in a greater generality. The main objective of our paper is to elaborate on how the method of Kalinin and Katok applies to the setting of Furstenberg's conjecture. A secondary objective is to discuss a topological analogue of Furstenberg's conjecture, namely, that the minimal sets of the action by the doubling and tripling maps on the circle is either finite or the circle itself.

Table of Contents

Chapter 1

Introduction

Throughout the paper, we use the following notations. K is the circle $\mathbb{R}/\mathbb{Z}$. We usually use the additive form $K \cong [0,1]/0{\sim}1$. $E_2 : K \to K$ is the doubling map $E_2(x) = 2x$, and $E_3 : K \to K$ is the tripling map $E_3(x) = 3x$. α is the $\mathbb{N}^2$ group action on K defined by $\alpha(n,0)(x) = E_2^n(x)$ and $\alpha(0,m)(x) = E_3^m(x)$. We say that a measure μ on K is α-*invariant* if and only if, for each $(n,m) \in \mathbb{N}^2$ and measurable $E \subset K$, we have $\mu(\alpha(n,m)^{-1}(E)) = \mu(E)$.

Furstenberg's conjecture, which is the focal point of the paper, is the following[1]: Lebesgue measure is the unique non-atomic α-invariant ergodic probability measure on K. Furstenberg proved that every minimal set is either finite or the whole space as a special case of his results in [2], which is a topological equivalent of the conjectured result. Rudolph [6] proved a partial result, namely, that the Lebesgue measure is the only such measure if the entropy of the measure is assumed to be positive.

This expository paper illustrates the method described by Kalinin and Katok in [3], which is used to prove the result of Rudolph in a greater generality. The paper by Kalinin and Katok also outlines how the method applies to our setting, and the main objective of our paper is to elaborate on this discussion. The secondary objective is to discuss the result that the minimal sets of the action is either finite or K, which is a topological analogue of Furstenberg's conjecture.

[1] For a history of the conjecture, see http://www.aimath.org/WWN/furstenburg/articles/html/3a/

Chapter 2

Preliminaries

2.1 Topological Dynamics

We denote by $\mathcal{O}(x)$ the orbit of $x \in K$, i.e. $\mathcal{O}(x) = \{\alpha(n,m)(x) \mid n, m \geq 0\}$. A set $E \subseteq K$ is called *invariant* if and only if $\alpha(n,m)(E) \subseteq E$ for any $n, m \geq 0$. $E \subseteq K$ is called *minimal* if and only if E does not contain any non-empty, proper, closed invariant subset under the action.

$x \in K$ is called a *recurrent point* for $f : K \to K$, if and only if for each $\epsilon > 0$ and $N \geq 0$, there exists $n > N$ such that $|f^n(x) - x| < \epsilon$.

See [4] for discussions of expansive maps and recurrent properties of topological dynamical systems.

2.2 Natural Extension

Given any measurable dynamical system $(M, \mathcal{B}, \mu, f)$, where μ is a probability measure, one can consider its natural extension $(\bar{M}, \bar{\mathcal{B}}, \bar{\mu}, \sigma)$. The phase space is given by $\bar{M} := \{(\omega_j) \in M^{\mathbb{N}} \mid f(\omega_j) = \omega_{j-1}\}$, i.e. ω_0 is the current state, and each ω_j for $j > 0$ is the j^{th} past. Cylinders

$$C(E_{i_1}, \cdots, E_{i_k}) := \{(\omega_j) \mid \omega_{i_1} \in E_{i_1}, \cdots, \omega_{i_k} \in E_{i_k}\}$$

generate $\bar{\mathcal{B}}$.

If M has a topology, the topology on $\bar{M}$ is generated by cylinders of open sets. If M has a metric d, it can be lifted to a metric on $\bar{d}$, and it is given by $\bar{d}(\omega, \eta) = \sum_{k=0}^{\infty} d(\omega_k, \eta_k)/2^k$.

Remark.

1. For each $x \in M$, $\sigma\restriction_{\pi^{-1}(x)}$ is uniformly contracting with respect to $\bar{d}$.
2. If Λ is a closed invariant set of M, then $\bar{\Lambda} = \pi^{-1}(\Lambda)$ is the unique closed invariant subset of $\bar{M}$ such that $\pi(\bar{\Lambda}) = \Lambda$.
3. If $\bar{\mu}$ is an invariant measure for σ, then $\pi_*\bar{\mu}$ is invariant for f.

Another notable property of a natural extension is the following, which is discussed in [7].

Proposition 2.2.1. Let M be a complete separable metric space, and $f : M \to M$ be continuous. Every invariant probability measure μ admits a unique lift $\bar{\mu}$ to $\bar{M}$. Furthermore,

$$\begin{aligned}\bar{\mu}(C(E_0, \cdots, E_k)) &= \bar{\mu}(C(f^{-k}(E_0) \cap \cdots \cap f^{-1}(E_{k-1}) \cap E_k)) \\ &= \mu(f^{-k}(E_0) \cap \cdots \cap f^{-1}(E_{k-1}) \cap E_k).\end{aligned}$$

2.3 Measurable Hull of a Partition

[1] contains a good discussion on ideas which we will need in our subsequent discussion, namely, measurable partition, measurable hull, and Pinsker partition. We summarize the discussion here.

Let ξ and η be partitions of a compact metric space X with a Borel measure μ. We do not require elements of a partition to be measurable. We denote by $\xi \vee \eta$ the *join* of ξ and η, which is defined by $\xi \vee \eta := \{U \cap V \mid U \in \xi, V \in \eta\}$. We say that ξ is *finer* than η if and only if, for each $U \in \xi$, there exists $V \in \eta$ such that $U \subseteq V$. We denote this relation by $\eta \leq \xi$. $\leq$ is a partial order of the collection of all partitions of X. We say that ξ and η are *equivalent mod 0* if and only if there exists a set $E \subseteq X$ of full measure such that $\{U \cap E \mid U \in \xi\} = \{V \cap E \mid V \in \eta\}$, and we write $\xi = \eta (\text{mod } 0)$.

Define $\mathcal{B}(\xi)$ to be the collection of all measurable subsets of X that are unions of elements of ξ. Let $\mathcal{T}$ be the Borel σ-algebra on X. Note that $d(A, B) = \mu(A \triangle B)$ defines a pseudo-metric on X, which, in turn, defines an equivalence relation on X by $A \sim B \iff d(A, B) = 0$. We say that $(X, \mathcal{T}, \mu)$ is *separable* if and only if $\mathcal{T}/\sim$ is separable as a metric space. For a separable measure space $(X, \mathcal{T}, \mu)$ and a $\mathcal{A} \subseteq \mathcal{T}$, a partition of ξ of X into measurable sets such that $\mathcal{A} = \mathcal{B}(\xi)$ (mod 0) can be constructed [1, p.9], and we denote this by $\Xi(\mathcal{A})$. We define the *measurable hull* of a partition ξ to be $\mathcal{H}(\xi) := \Xi(\mathcal{B}(\xi))$. We say that a partition is *measurable* if and only if it is equivalent mod 0 to its measurable hull. The measurable hull is the finest measurable partition which coarsens ξ.

Let $T : X \to X$ be a measure-preserving map with a measure-preserving inverse, and ξ be a partition consisting of measurable sets. Define

$$\xi^- := \bigvee_{n=0}^{\infty} T^{-n}\xi.$$

ξ^- is a measurable partition, and has the property that, if ξ^- is invariant (i.e. $T^{-1}\xi^- = \xi^-$) then ξ has zero entropy, and otherwise, ξ has positive entropy [1, p.21]. Next, define

$$\Pi(\xi) := \mathcal{H}\left(\bigwedge_{n=1}^{\infty} T^{-n}\xi^-\right),$$

where $\xi \wedge \eta$ is defined as the finest partition that coarsens both ξ and η. For $\Pi(\xi)$, we have the following: if $\eta \leq \Pi(\xi)$ is a finite of countable partition with finite entropy, then the entropy of η is zero [1, p.22]. Now, consider all partitions with such property. The collection is partially ordered by $\leq$, so the finest partition such that any coarser finite partition has zero entropy exists. We call the partition the *Pinsker partition*, and denote it by $\pi(T, \xi)$. An important fact is that the Pinsker partition is the measurable hull of the partition into global unstable manifolds [1, p.27], which appears in the discussion of the main theorem.

Chapter 3

Minimal Sets of The Action

Proposition 3.0.1. The orbit of a rational point is periodic or eventually periodic. The periodic orbits are $\{0\}$ and the orbit of $\frac{p}{q}$ where $(q, 2) = (q, 3) = (p, q) = 1$

Proof. This is easy to check. □

Lemma 3.0.2. 1 is an accumulation point of $\{2^n 3^m (\mathrm{mod}\ 1) \mid n, m \in \mathbb{Z}\}$.

Proof. Note that the orbit of 3^{-k} consists of all points of the form $\frac{q}{3^k}$ where $(q, 3) = 1$. Since $(3, 3^k - 1) = 1$, we have $\frac{3^k-1}{3^k} \in \left\{2^n 3^{-k} (\mathrm{mod}\ 1) \mid n \geq 0\right\}$ for any $k \geq 0$. It follows that $\frac{3^k-1}{3^k} \in \{2^n 3^m (\mathrm{mod}\ 1) \mid n, m \in \mathbb{Z}\}$ for any $k \geq 0$. □

Theorem 3.0.3. The orbit of an irrational point is dense.

Proof. Fix $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Suppose $\mathcal{O}(\alpha)$ is not dense. Note that $W := \overline{\mathcal{O}(\alpha)}$ has an empty interior, since otherwise $W = K$ by invariance of W. Hence, every point of W is a boundary point. Since W is compact, the set of recurrent points R of $E_3\restriction_W$ is non-empty. Fix $x \in R$. Since $K \backslash W$ consists of open intervals, x is contained in the boundary of some open interval I. Without loss of generality, we may assume that x is the right endpoint of I. Suppose that the length of I is l.

By the preceding lemma, there exists $n, m > 0$ such that $x - l < \frac{2^n}{3^m} x < x$. Let $\epsilon := \frac{2^n}{3^m}$. Since x is a recurrent point, there exists $M > m$ such that $\left|3^M(x) - x\right| < \min\left(\frac{1-\epsilon}{\epsilon} x, l\right)$. Then, we have $2^n 3^{M-m} x \in I$, which contradicts the fact that W is invariant.

Corollary 3.0.4. Every closed invariant set of the action is either K or finite.

Chapter 4

Main Theorem

In this section, we follow the discussions in sections 2.1-2.3 of [3]. Let μ be a positive-entropy ergodic invariant measure on (K, α). Our goal is to show that μ is the Lebesgue measure. Sections 2.1 and 2.2 describes the method for Cartan actions on the three-dimensional torus. Section 2.3 outlines how the method can be adopted to our setting.

4.1 The Structure of the Natural Extension

The natural extension of our action on K, which we denote by S, is the dual group to the discrete group $\mathbb{Z}(1/2, 1/3)$, and it is isomorphic to a solenoid [5, Appendix]. S can be thought of as a subset of $K^{\mathbb{Z}^2}$; more precisely, each point $(\omega_{i,j})$ has the form

$$\begin{array}{cccc} \omega_{0,0} & \omega_{1,0} & \omega_{2,0} & \cdots \\ \omega_{0,1} & \omega_{1,1} & \cdots & \\ \omega_{0,2} & \vdots & \ddots & \\ \vdots & & & \end{array}$$

where $E_2(\omega_{i,j}) = \omega_{i-1,j}$ and $E_3(\omega_{i,j}) = \omega_{i,j-1}$. The maps E_2 and E_3 extend to shifts on $K^{\mathbb{Z}^2}$, which we denote by σ_2 and σ_3, respectively. We have

$$\sigma_2 : \begin{array}{cccc} \omega_{0,0} & \omega_{1,0} & \omega_{2,0} & \cdots \\ \omega_{0,1} & \omega_{1,1} & \cdots & \\ \omega_{0,2} & \vdots & \ddots & \\ \vdots & & & \end{array} \quad \mapsto \quad \begin{array}{cccc} E_2(\omega_{0,0}) & \omega_{0,0} & \omega_{1,0} & \cdots \\ E_2(\omega_{0,1}) & \omega_{0,1} & \cdots & \\ E_2(\omega_{0,2}) & \vdots & \ddots & \\ \vdots & & & \end{array}$$

and

$$\sigma_3 : \begin{array}{cccc} \omega_{0,0} & \omega_{1,0} & \omega_{2,0} & \cdots \\ \omega_{0,1} & \omega_{1,1} & \cdots & \\ \omega_{0,2} & \vdots & \ddots & \\ \vdots & & & \end{array} \mapsto \begin{array}{cccc} E_3(\omega_{0,0}) & E_3(\omega_{1,0}) & E_3(\omega_{2,0}) & \cdots \\ \omega_{0,0} & \omega_{1,0} & \cdots & \\ \omega_{0,1} & \vdots & \ddots & \\ \vdots & & & \end{array}$$

Thus, the action on K extends to S. We denote this action by α, as we will no longer work with (K, α), and there will be no confusion. $\pi : S \to K$ is defined by $\omega \mapsto \omega_{0,0}$. By Proposition 2.1, μ lifts to a measure that is invariant with respect to the action on S. We denote this measure by $\bar{\mu}$.

Note that once we fix $\omega_{0,0}$, $\omega_{i,0}$ for $i \geq 1$, and $\omega_{0,j}$ for $j \geq 1$, $\omega_{i,j}$ for each $i, j \geq 0$ is determined. For each $\omega_{0,0}$, there are two possible values of $\omega_{1,0}$ and three possible values of $\omega_{0,1}$. The same observation holds for $\omega_{i,0}$ and $\omega_{0,j}$ for each $i, j \geq 1$, as well. Note that σ_2 is a contraction on S with respect to the usual metric, and its Lipschitz constant is $1/2$, and the Lipshitz constant of σ_3 is $1/3$. Hence, $\pi : S \to K$ can be thought of as a fiber bundle with $\mathbb{Z}_2 \times \mathbb{Z}_3$ as fibers, where $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are 2- and 3-adic integers respectively. Recall that a 2-adic integer a can be represented as a sequence $(a_0, a_1, \cdots)$ via the power series expansion $a = \sum_{i=0}^{\infty} a_i 2^i$, where $a_i \in \{0, 1\}$. The $\mathbb{Z}_2$ coordinate encodes the past of p with respect to E_2 The same observation holds for the $\mathbb{Z}_3$ coordinate.

4.2 Suspension Construction

Following the discussion in [3, Section 1.2.2], we obtain an $\mathbb{R}^2$-action starting from our $\mathbb{Z}^2$-action on S. We refer to the object that $\mathbb{R}^2$ acts on as S, and the reason is as follows. First, the $\mathbb{R}^2$-action is similar to the $\mathbb{Z}^2$-action on S, and second, we will focus on the $\mathbb{R}^2$-action instead of the $\mathbb{Z}^2$-action from now on. We think of a line in $\mathbb{R}^2$ as a "time direction" of the action. The construction allows us to speak of "acting in an irrational time direction," as we do in the following discussion.

4.3 Real Foliation W of S

Recall that each point in S has the form

$$\begin{array}{cccc} \omega_{0,0} & \omega_{1,0} & \omega_{2,0} & \cdots \\ \omega_{0,1} & \omega_{1,1} & \cdots & \\ \omega_{0,2} & \vdots & \ddots & \\ \vdots & & & \end{array}$$

with restrictions $E_2(\omega_{i,j}) = \omega_{i-1,j}$ and $E_3(\omega_{i,j}) = \omega_{i,j-1}$. Now, consider perturbing $\omega_{0,0}$, and observe how it affects the past states. It is easy to see that when $\omega_{0,0} \mapsto \omega_{0,0} + \epsilon$, we have $\omega_{1,0} \mapsto \omega_{1,0} + \epsilon/2$, $\omega_{0,1} \mapsto \omega_{0,1} + \epsilon/3$, and so on for the rest of the past states. By varying the amount of purturbation ϵ, we obtain a leaf going through $(\omega_{i,j})$. A collection of these leaves glued together gives a foliation on S, which we denote by W. We refer to W as the real foliation of S.

4.4 Critical Time Direction for W

Given two points on W, the action by $(1, 0)$ doubles the distance between the two, and the action by $(0, 1)$ triples it. Then, there is some $-1 < y < 0$ such that the action by $(1, y)$ is an isometry on W. The purpose of this section is to find such y by computing the Lyapunov exponents of the action. See [5, Appendix] for a discussion on Lyapunov exponents for maps on p-adic integers.

For each $x \in K$, there is a small neighborhood $U \ni x$ such that $\pi^{-1}(U)$ can be identified with $U \times \mathbb{Z}_2 \times \mathbb{Z}_3$. We let $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$ play the role of a tangent space at a point in $U \times \mathbb{Z}_2 \times \mathbb{Z}_3$. We equip $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$ with the product norm of the usual norms on $\mathbb{R}$, $\mathbb{Z}_2$, and $\mathbb{Z}_3$. Then, the Lyapunov exponent of the action by $(x, y) \in \mathbb{R}^2$ in the real direction is

$$\begin{aligned}
\lambda(x, y) &= \lim_{m\to\infty} \frac{\log ||(x, y)^m.(1, 0, 0)||}{m} \\
&= \lim_{m\to\infty} \frac{\log ||(2^{xm}3^{ym}, 0, 0)||}{m} \\
&= \lim_{m\to\infty} \frac{xm \log 2 + ym \log 3}{m} \\
&= x \log 2 + y \log 3.
\end{aligned}$$

Hence, the action by each point in the line $\{(x, y) \in \mathbb{R}^2 \mid x \log 2 + y \log 3 = 0\}$ has zero Lyapunov exponent. We call this time direction the *critical direction* for W.

4.5 The ergodic components of the action in the critical direction

First, we check that the action in the critical direction is ergodic. We already know that the action by (n, m) for any $n, m \in \mathbb{Z}$ is ergodic, but not for any $(x, y) \in \mathbb{R}^2$. To prove the ergodicity of the action in the critical direction, we show that each ergodic component of the action in the critical direction is a union of leaves of W. We follow the argument in [3][Section 2.2.3].

Let a be a non-zero element in the critical direction, such as $a = (1, -\log 2/\log 3)$, and let ξ_a be the partition into ergodic components of a. Let $\mathcal{H}(W)$ be the measurable hull of the partition into the leaves of W.

Let W' be the one-dimensional stable foliation of a. The Birkhoff sum of any continuous function on a stable leaf is constant, so every stable leaf of a is contained in an ergodic component of a. Hence, we have $\xi_a \leq W'$. Since $\mathcal{H}(W')$ is the finest measurable partition which coarsens W', we have $\xi_a \leq \mathcal{H}(W')$.

Let W'' be the unstable foliation of a. Let b be an element not in the critical direction such that W' is its stable foliation, and $W \oplus W''$ is its unstable foliation. We may obtain this element by slightly moving a off the critical direction. In our example, we may take $b = (1, -\log 2/\log 3 - \eta)$, where $\eta > 0$ is small. Now, we use the fact that both the measurable hulls of partitions into leaves of the stable and unstable foliation generate the Pinsker σ-algebra (see [1, p.27]). By applying this fact to b, we obtain $\mathcal{H}(W') = \mathcal{H}(W \oplus W'')$.

Since we have $\mathcal{H}(W \oplus W'') \leq W \oplus W'' \leq W$ and $\mathcal{H}(W)$ is the finest coarsening of W, we have $\mathcal{H}(W \oplus W'') \leq \mathcal{H}(W)$. Combining the results thus far, we obtain $\xi_a \leq \mathcal{H}(W)$, as desired.

4.6 Conclusion

Following [3][Section 2.2.2], we see that, for almost every leaf L of the foliation W, the conditional measure μ_L is invariant under the set of translations of full μ_L measure. Then, the arguments in [3][Section 2.2.1] give our desired result. The arguments in 2.2.1 and 2.2.2 are general, and no remark needs be added for our special case.

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